{ } **LESSON**

Infinite Recess

by Emily Clader and Daniel Reck, 826michigan

GRADES 6–7

TYPE
NARRATIVE
POETRY
STEM

COMMITMENT
2 SESSIONS: 2 HOURS EACH

We're letting you in on a bit of a secret here: math and writing are pretty much the same thing. This might come as a surprise, but think about it. Like writing, math helps us understand the world by allowing us to describe the rules it follows and explore what would change if we broke those rules. In this lesson, students learn how imposing structure on writing can inspire them even while constraining them. By developing an awareness of mathematical structure in life, they will be able to imagine the world in a different and exciting new way. Together, you'll create infinite poems using fractals, imagine life on a doughnut, and speculate about a universe where time goes crazy. Along the way, everyone will learn some cool math facts and some helpful writing strategies.

SESSION 1: FRACTAL POETRY

In this session, students learn that shapes don't have to be silent, and poetry doesn't have to be linear, as they write shapes in verse, and verse in shapes.

YOU WILL NEED

- Copies of the “Fractal Poetry: A Step-by-Step Guide” handout

BEFORE YOU START

A Note on the Sections Titled “FOR YOU TO KNOW (AND YOUR STUDENTS TO DISCOVER)”

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This section is for you—and not the students. You shouldn’t feel like you have to read it, and you certainly shouldn’t feel like you have to teach it. If you do choose to teach this material to students, we’ve found it’s best not to introduce these ideas or vocabulary words at the beginning of a lesson. As a lesson progresses and students are sharing their observations and ideas, it will usually become clear that they have discovered many of these concepts for themselves. You can then help them by providing labels (the vocabulary words) to hang on the scientific ideas they have already described in their observations. As you do this, try to draw on students’ own language in connecting their observations and ideas to the “scientific” language. If, at the end of the lesson, your students haven’t learned what they were “supposed” to, resist the urge to give it to them. Their questions and curiosity will do a better job of motivating learning over the long term. Waiting is powerful—and you will see the results in what your students do and learn.

HOW TO BEGIN (20 MINUTES)

The Lesson Takes Shape

Start by asking students to share their favorite shape and why they like it. A triangle? A rhombus? Will someone go wild and choose a dodecagon? Once everyone’s had a chance to voice his or her polygon preferences, dive into some meatier topics:

- What is math? What does it do, or what is it about? Why do you do math?
- Does math have anything in common with writing? What about creative writing?

After allowing the students to share their ideas on these questions, offer a definition of math, such as the following:

Math is a *language* in which we can give precise definitions of concepts that would otherwise be hard to pin down.

STEP 1 (10 MINUTES)

Multidimensional Math

Your students probably have some idea of what dimensions are. They know that a sheet of paper is two-dimensional, or that the planet Earth is three-dimensional. But what about a collection of dots (or points, mathematically speaking)—does that have a dimension? In the language of math, it’s possible to give a definition of dimension. Then, equipped with that definition, we suddenly have the power to make sense of new ideas—we can figure out what dimension that collection of dots has. And we can know what four-dimensional means, or one hundred–dimensional, or even one half–dimensional or infinity-dimensional. (Check out this lesson's "For You to Know" section for some helpful explanations related to this concept, as well as a few other mathematical ideas that come up in the activities.)

When viewed as a language, mathematics becomes a tool both for describing the world and for expanding the scope of our imagination. In this way, it is similar to creative writing. A mathematician, like a writer, uses the familiar as a springboard to conceive of new worlds, and gives those worlds substance by putting them into language.

STEP 2 (30 MINUTES)

Zooming In

In this session, students will learn about a mathematical object known as a **fractal.** Before you explain exactly what a fractal is, it's helpful to show a few pictures to spark imagination and curiosity. (There are a ton of great images to choose from online; just search for "fractal" and select a handful of eye-catching examples, ideally looking for ones that clearly illustrate the definition.) You can also show students a video. Search online for "fractal zoom," which will yield some stunning videos of fractals in motion.

Allow the students a moment to try to articulate what these images have in common, or, more simply: What's so cool about these pictures and videos? After a short discussion, define the term:

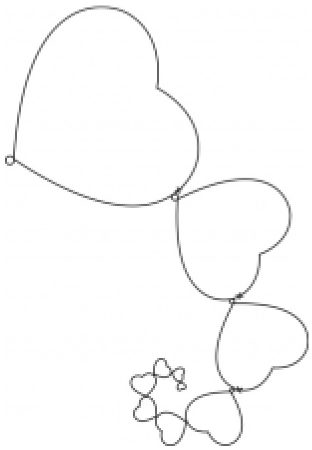
> A **fractal** is a shape that contains a copy of itself *inside of itself*. If you zoom in on one part of the shape and keep zooming in, you'll eventually see something that looks just like what you started with.

We've found it's helpful to return to the images while explaining this concept to students, pointing out the smaller copies of the fractal inside the larger shape. There are even examples of fractals in nature, like Romanesco broccoli, where each floret is made up of smaller florets that each look like a whole piece of broccoli. Some simple fractals, like the Sierpinski triangle or the Koch snowflake, are easy to draw on your own. (We highly recommend that you have students try to draw one or the other; there are instructions in the “For You to Know” section of this lesson, so you can teach the students how to draw them if time and interest permit.)

We’ve come up with a simple method for turning *any* shape into a fractal. It will take students fifteen minutes the first time, give or take. Here’s the method:

1. Draw a simple shape.
2. Pick any two points on the perimeter of the shape. Label one of the two points with an *X* and the other point with an *O*.
3. Draw a smaller copy of the shape (about half the size of the original one), with its own *X* and *O* in the same places as on the original shape. But here’s the key: draw it so the *X* on the smaller shape is *on top of* the *O* on the original shape. This will probably require you to rotate the smaller shape, but that’s good.
4. Draw an even smaller copy of the shape (about half the size of the previous one), with its *O* on top of the *X* on the previous shape.
5. Repeat until the shapes get too small to draw.

It’s a bit tricky to explain the procedure in words, but an example should make the idea clear:



Model the procedure for your students first. Then ask them to try it for themselves with a very simple shape to get the hang of the method. If your students have no problems with triangles, circles, or hearts, see what they can do with smiley faces, airplanes, or trees! Be sure to leave time for sharing.

STEP 3 (25 MINUTES)

Zooming In

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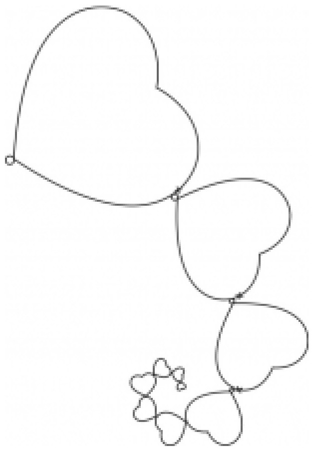

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STEP 4 (20 MINUTES)

Get That Writing in (a) Shape

With the rest of the handout, you can show your students how to turn their recursive poem into a fractal. First (step 2 on the handout), they pick a shape related to the theme or idea of their poem. This is going to be the basis for a fractal. They'll need to draw it a lot of times, so they shouldn't be afraid to keep it very simple. (Of course, don't discourage ambition.) Students can use the back of the handout or a piece of scratch paper to practice drawing their shape until they've figured out something they like and can draw easily.

For step 3, students should find a way to arrange their poem inside the shape. It can go through the inside or around the edge, in lines or in crazy loops. Here's how a poem might look if it were arranged inside a speech bubble, as imagined by 826michigan student Angela Zhang, age eleven, in her poem "Once Upon a Time":

> **Insert Graphic [c11uf002.eps]**

Finally (step 4 on the handout), students make their shape and poetry into a fractal! To do this, they'll use the procedure you practiced together earlier. First, they will mark the spot where the poem begins with an *O* and the spot where the poem ends with an *X*. Then, as before, they will make smaller and smaller copies of the shape, with the *O* of each new copy sitting on top of the *X* of the previous copy. In each shape, they will write their poem, so the end of the poem on one shape flows directly into the beginning of the poem on the next shape.

STEP 5 (15 MINUTES)

From Beginning to End and Back Again

At the very end of the session, ask everyone to share and marvel at the results.

SESSION 2: LEAVING SPACE AND TIME

In this session, students will use ideas from mathematics to imagine and write about worlds in which space and time behave strangely. The first half of the session will be devoted to space, leaving the second half to think about time.

YOU WILL NEED

- Modeling clay or the like
- Foil
- Copies of the "Leaving Space and Time" handout
- String, pipe cleaners, or long strips of paper

BEFORE YOU START

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HOW TO BEGIN (30 MINUTES)

Playing with Shapes

For playing with space, the most important mathematical tool is the subject of **geometry.** Once again, start the class by asking students to share their opinions. Ask them:

- What is geometry?
- If you think geometry has something to do with shapes … what's a shape?

These are hard questions with no single right answer, but one combined definition might be as follows:

A **shape** is a collection of points in space, and **geometry** is the study of properties of shapes.

For example, some common properties of shapes that a geometer (a mathematician who studies geometry) might study are length, area, angle, and dimension.

Mathematicians sometimes talk about shapes as abstract things, like "a circle of radius 2" and describe their properties without having any particular circle in front of them. But you can also think about the geometry of the world around you: What is the geometry of our world like?

To get students started, you might note that Earth is a sphere (roughly, but with mountains and valleys). Ask students to think about some of the specifics of this geometry, such as the size of that sphere (very big), and the dimension of the space we move around in (three, but stuck to a two-dimensional ground). They might even notice the fact that Earth is a finite thing, whereas outer space might very well go on and on forever.

There are many stories about characters who live in worlds whose geometry is very different from that of our own. Putting a character in a different sort of space is a way to present him or her with challenges (and sometimes, opportunities) that would not occur on Earth. In Antoine de Saint-Exupéry's *The Little Prince*, for example, we learn of a boy who lives on a very small planet, who worries that a few trees will overtake the entire surface of his world.

Size is actually a minor difference in geometry; there are much bigger ones. To get students thinking about different geometries, give them some modeling clay, and have them make a planet in a fun shape. Let them know that some of the craziest math is in the field of topology—a cousin of geometry. In topology, a circle and a square are the same thing because, roughly speaking, you can stretch and bend one into the shape of the other. However, a doughnut (mathematically called a torus) is not the same as a ball; you would have to squish the hole in the middle of the doughnut, and that's not allowed. Topologically you can bend and stretch one shape into another, but you're not allowed to close holes or cut out new ones. (Remember from the first session that definitions and rules are important!) Ask your students if they can stretch their shape into a ball—if so, topologically it's the same as the planet Earth. Challenge your students to create a shape that's topologically different from a ball, and then as many topologically different shapes as they can!

Edwin Abbott Abbott's *Flatland* (available online) offers another classic example of interesting geometry; this late-nineteenth-century novella describes a two-dimensional world where the inhabitants are all objects like lines and triangles and squares. Among the many difficulties faced by these two-dimensional inhabitants is the challenge of recognizing each other from afar, because any two-dimensional object looks just like a line segment when viewed head-on.

After they wrestle with that for a bit, bring up Vi Hart's *Wind and Mr. Ug*. Ask your students what's going on! (The quick answer is that Wind *is* Mr. Ug! Wind is on a Möbius strip, a special shape that has only one side.) It's fine if they don't understand; let their curiosity and wonder linger. If they do understand that they're seeing a Möbius strip, you can let them know that there's another crazy shape with only one side, the Klein bottle. It can only exist in four dimensions, and its inside is also its outside!

We've discussed a number of examples of weird geometry in fiction, but there's another great source of interesting geometries: video games. The Mario games have some awesome geometry; *Super Mario Bros*. and many of its sequels are great two-dimensional fun, and *Super Mario Galaxy* and *Super Mario 3D World* have amazing three-dimensional mechanics. (*Super Paper Mario* actually requires switching between 2-D and 3-D views of the world; if you can show a video of this, it's a really great example of how geometry and perspective are connected.) The game *Portal* shows you what might happen if jumping through one point in space suddenly put you at a different point somewhere entirely different. Ask students if they can think of other examples of video games (or examples from fiction that were missed earlier) in which geometry works differently from the way it does in our world.

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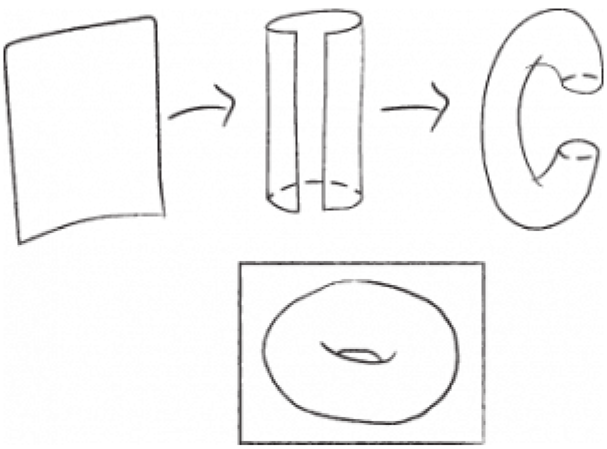

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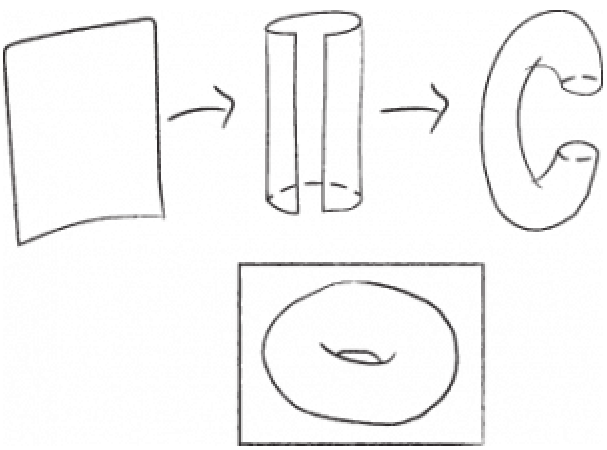

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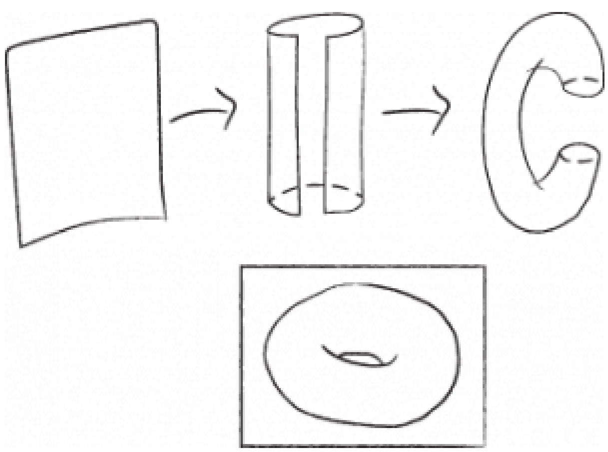

STEP 3 (20 MINUTES)

Time and Space to Write and Reflect

Now, ask students to pick one or both of the two ideas generated earlier on their handout, about a different kind of space or time, and flesh out as much of a story as possible, using the third part of the handout. Try to save enough time—five or ten minutes—to allow a few of your students to share some of their work aloud.

To end, after students have shared, conclude the lesson by revisiting the question with which the lesson began, What is math? Ask students whether their answer to this question changed over the course of the two sessions.

We think math is awesome. Students often think of math as a collection of facts and methods for solving math problems. Formulas are extremely important for you to be able to understand and use, but there's so much more to math! The formulas and methods students learn in class have not always been known by human beings. They are the fruits of countless hours of labor by people trying to figure out how the world works, in many cases thousands of years ago. These people imagined the world in tons of different ways until they stumbled on the right rules to describe it (and many other worlds besides). And mathematicians are still learning more all the time!

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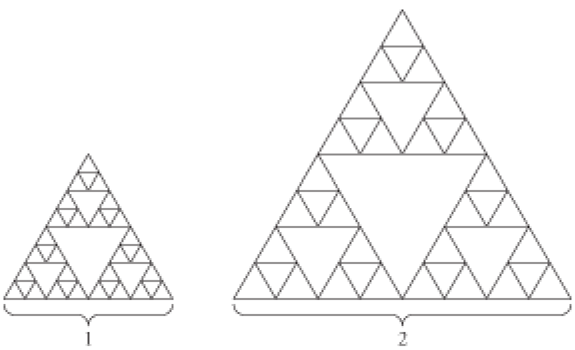

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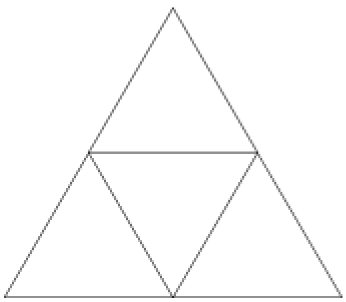

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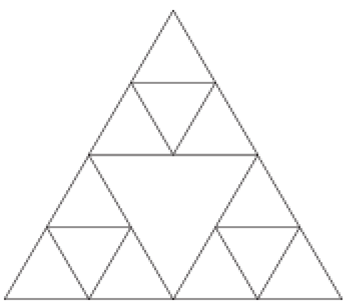
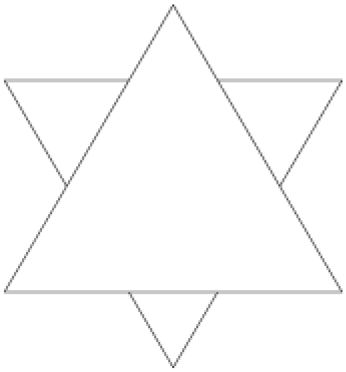


Repeat with each of the nine triangles that result, and so on.

The Koch snowflake also starts with a triangle. In the middle of each of its edges, draw a smaller triangle jutting out:



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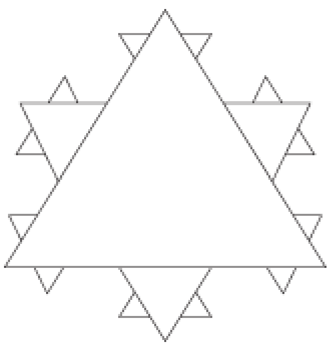

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As mentioned in the lesson, recursion occurs when something feeds back into itself. An example would be two identical mirrors positioned exactly across from each other, reflecting each other into infinity. Another example would be the box of cocoa that depicts a model holding the same box of cocoa, and so on, and so on. Fractals are also recursive; when you zoom in, you're led back where you started.

Topology

We'll say it: topology is crazy (cool), if a little mind blowing. Topology is similar to geometry, only geometry looks at rigid things (angles, lengths, area, volume, and so on), whereas topology looks at things that can be stretched, twisted, bent, and squeezed—but not torn.

For an example, think of a coffee cup and a doughnut. If you make the cup's handle fatter and its bottom thicker, eventually it would look like a doughnut. And although it would be different in some ways, in others, it would be the same—for instance, it would still have a path, or a "tour," around the hole in the middle.

On a sphere, any tour that returns to its starting point can be retraced with shorter and shorter ones, until it shrinks to a point. But on the cup/doughnut, the tour around the hole is "noncontractible"—if you try to shorten it, you'll get stuck at some point, when it goes around the hole as tightly as possible.

You can also think of a tour as a closed loop; so a tour on a sphere divides it into a section inside the tour, and a section outside the tour (although you're free to decide which is which!). But the tour around the hole or handle of the cup/doughnut doesn't divide the surface into two parts—you can get from one side to the other without crossing it.

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STEP 4 (35 MINUTES)

Time and Space to Write and Reflect

Now, ask students to pick one or both of the two ideas generated earlier on their handout, about a different kind of space or time, and flesh out as much of a story as possible, using the third part of the handout. Try to save enough time—five or ten minutes—to allow a few of your students to share some of their work aloud.

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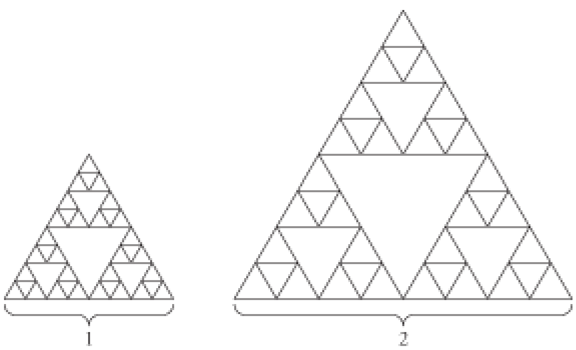

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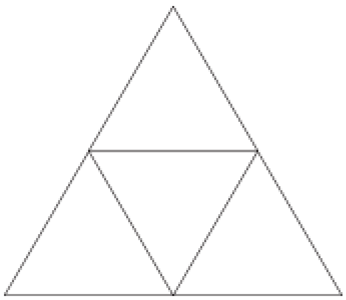

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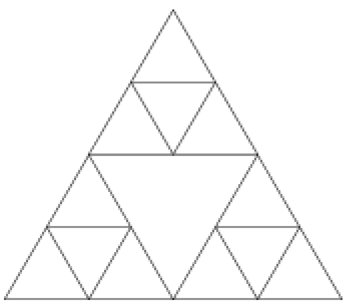

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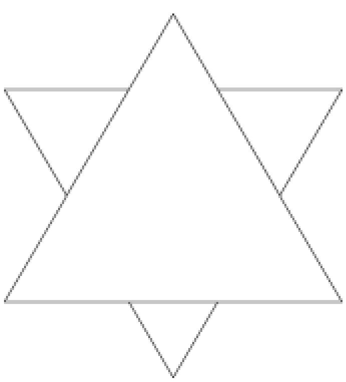

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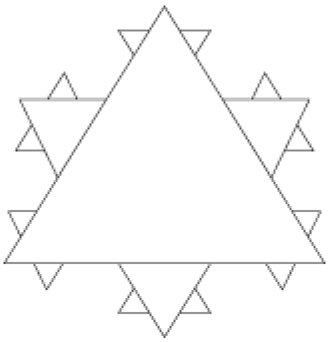

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